Beam estimation

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1 Direct estimation

To prevent the use of fitting methods to estimate the beam size, the main idea is to consider that (u, v) points were drawn from a 2D normal distribution. The beam shape is extracted from the empirical covariance of this distribution.

We have N points in the uv plane described by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ (including the null frequency $(u, v) = (0, 0)$). We can define the modulation transfer function h of the interferometer as:

$$\hat{h}(u, v) = \begin{cases} 1 & \text{if } u \in \boldsymbol{u} \text{ and } v \in \boldsymbol{v}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The dirty beam is h the Fourier transform of $\widehat{h}$. One way to define the resolution of the interferometer is to compute the so called beam as half-width at half-maximum of a 2D Gaussian g fitted on the central peak of the dirty beam. The covariance matrix of the Gaussian $\mathbf{C}$ are given by

$$\theta = \arg\min_{\mathbf{C}} \|g(\mathbf{C}) - h\|^2 \tag{2}$$

From Parseval-Plancherel theorem, this can be also written in Fourier domain as

$$\theta = \arg\min_{\mathbf{C}} \left\|\widehat{g}(\mathbf{C}) - \widehat{h}\right\|^2 \tag{3}$$

where the $\widehat{g}(\mathbf{C})$ is also a Gaussian of covariance matrix $\mathbf{D} = \pi^{-2}\,\mathbf{C}^{-1}$.

This covariance matrix $\mathbf{D}$ can be approximated from the distribution of the sampling point in the uv plane. As by construction $\langle \boldsymbol{u} \rangle = 0$ and $\langle \boldsymbol{v} \rangle = 0$ we have:

$$\mathbf{D} = \frac{2}{2N+1} \begin{pmatrix} \sum_n u_n^2 & \sum_k u_k\, v_k \\ \sum_k u_k\, v_k & \sum_n v_n^2 \end{pmatrix} \tag{4}$$

The $\frac{2}{2N+1}$ factor is to count twice all the (uv) points (due to the symmetry of the uv plane) excepted the null frequency. In addition to its covariance matrix $\mathbf{C}$, we can describe the beam with its principal angle θ and its half-widths at half maximum r_1 and

r_2 along both the major and minor axes respectively. These parameters can be extracted by the mean of the eigendecomposition:

$$\mathbf{D} = \mathbf{Q}\,\boldsymbol{\Lambda}\,\mathbf{Q}^{\mathrm{T}} \tag{5}$$

where $\mathbf{Q}$ and $\boldsymbol{\Lambda}$ are the matrices of eigenvectors and eigenvalues respectively. The matrix $\boldsymbol{\Lambda}$ is a diagonal matrix that contains the eigenvalues (λ_1, λ_2) that are the variances along both axis. $\mathbf{Q}$ is a rotation by the principal angle. As $\mathbf{C} = \pi^{-2}\mathbf{D}^{-1}$, its eigenvalues are $\boldsymbol{\Lambda}^{-1}$ and major and minor axis are inverted leading to a rotation of $\pi/2$ of the principal angle. As a result, these matrices can be expressed as a function of the beam parameters using as :

$$\mathbf{Q} = \begin{pmatrix} \cos(\pi/2-\theta) & -\sin(\pi/2-\theta) \\ \sin(\pi/2-\theta) & \cos(\pi/2-\theta) \end{pmatrix} \tag{6}$$

$$\boldsymbol{\Lambda} = \frac{\log(2)}{2\pi^2}\begin{pmatrix} r_1^{-2} & 0 \\ 0 & r_2^{-2} \end{pmatrix} \tag{7}$$

$$\tag{8}$$

where $\frac{\sqrt{2\log(2)}}{2}$ is a factor to convert standard deviation to half-width at half-maximum. The beam ellipse parameters are then:

$$r_1 = \frac{\sqrt{2\log(2)}}{2\pi}\frac{1}{\sqrt{\lambda_1}} \tag{9}$$

$$r_2 = \frac{\sqrt{2\log(2)}}{2\pi}\frac{1}{\sqrt{\lambda_2}} \tag{10}$$

$$\theta = \arctan\left(Q[1,1], Q[2,1]\right) \tag{11}$$

In OImaging, we can plot the beam by applying the transformation matrix $\mathbf{T} = \frac{1}{\pi}\,\mathbf{Q}\,\boldsymbol{\Lambda}^{-1/2}$ to a circle of diameter 1.

2 Results

See notebook `https://jovian.com/ferreols/beamexample`