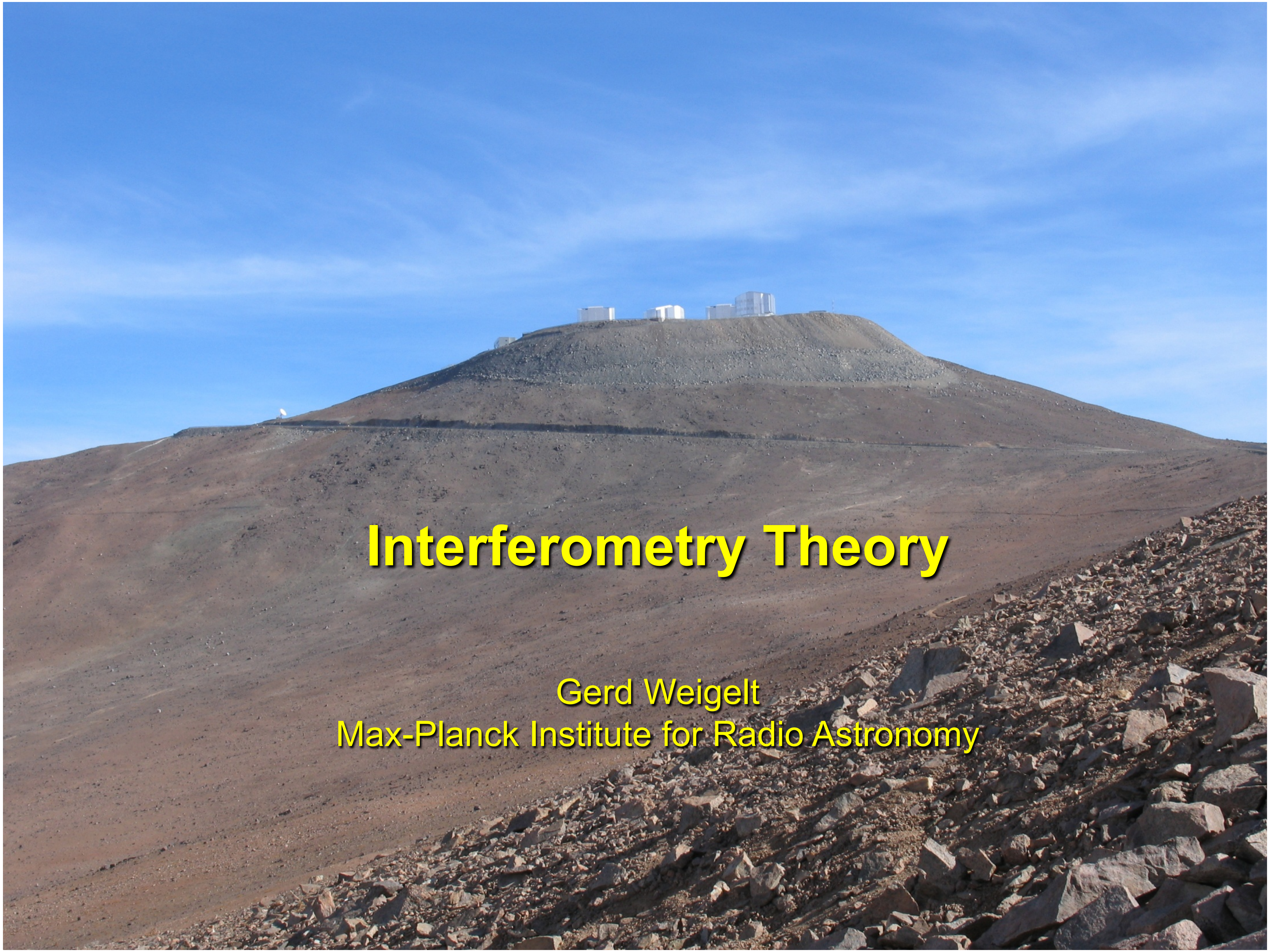

Interferometry Theory

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Plan

Introduction:

Diffraction-limited point spread function (psf)

Atmospheric wave front degradation and atmosperic point spread functions (speckle movie)

Space-invariance of psfs

2-telescope interferograms

Spectrally dispersed IOTA and AMBER-VLTI interferograms (IOTA movie)

LBTI interferograms

Optical experiments: Simulation of interferograms using halogen lamps and diffraction masks to observe psfs, image degradation, and interferograms

Simple principle of interferometry

Fourier transform properties of lenses

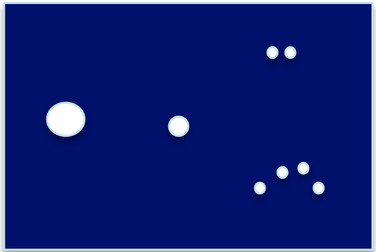
Calculation of the intensity distribution of interferograms and the resolution of interferometers

Wave optics (Fourier optics), incoherent imaging equation

Interferogram equation, Jennison's closure phase method, Van Cittert-Zernike theorem

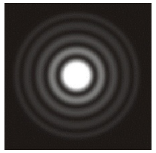
Optical experiments: Simulation of diffraction and atmospheric image degradation using halogen lamps, diffraction masks, and atmosphere simulator

What can we learn from these experiments? Looking through the various mask apertures, you can see white-light diffraction and interference pattern (telescope, interferometer, atmospheric image degradation, coherence time etc.)

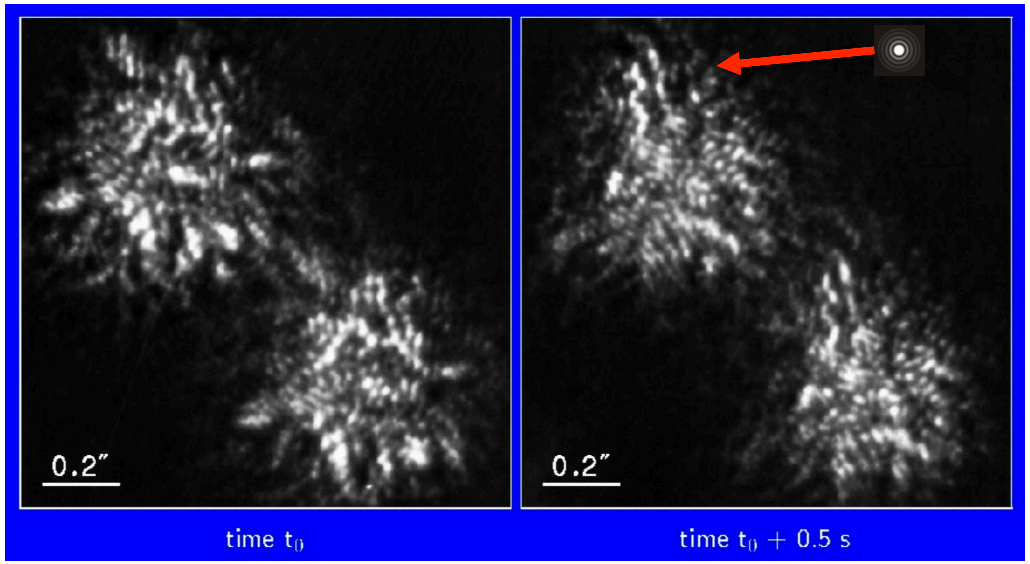


Diffraction apertures + atmosphere simulator to generate interferograms

Diffraction-limited point spread function of a single telescope

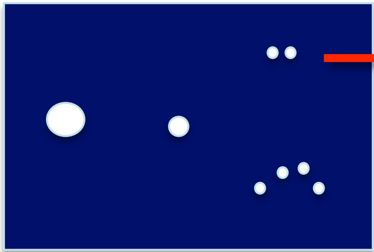


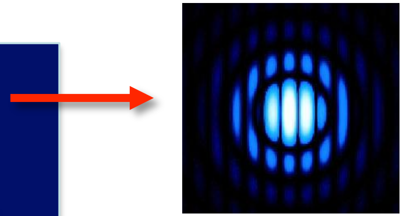
Space-invariance of the atmospheric psf



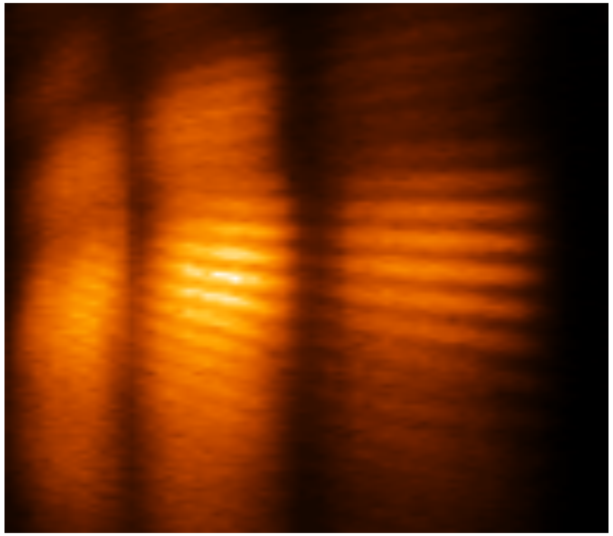
Optical experiments: Simulation of diffraction, telescopes and atmospheric image degradation using halogen lamps and diffraction masks

Optical experiment:
2-telescope interferogram
(without spectral dispersion)





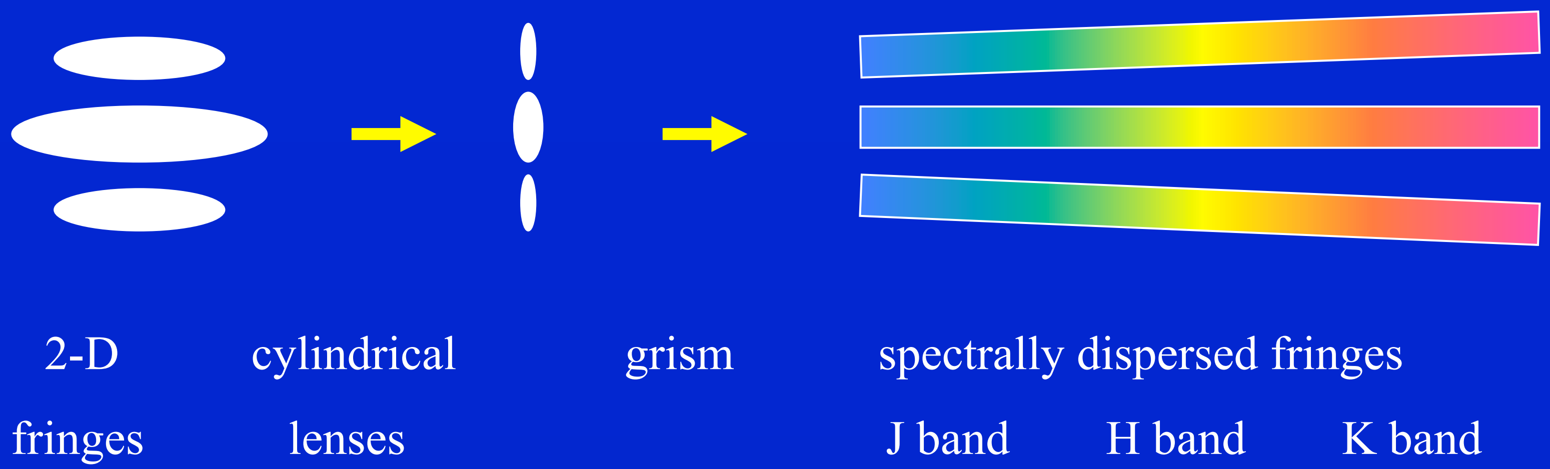
Spectrally dispersed IOTA interferograms:
Fringe motion caused by the atmosphere;
wavelength dependence of interferograms



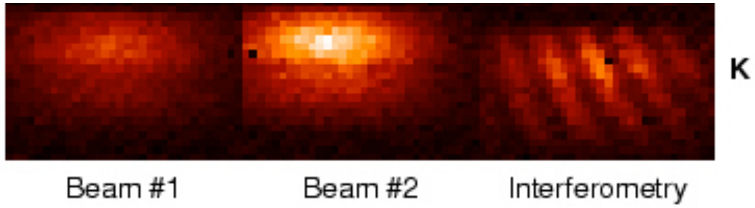
See movie at http://www.mpifr-bonn.mpg.de/153017/beobachtungen

IOTA JHK-band beam combiner instrument

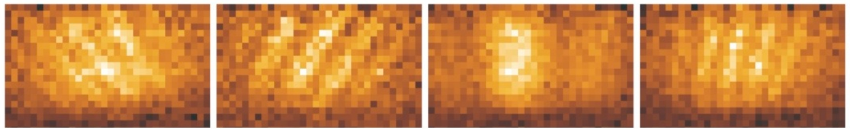
- Simultaneous recording of spectrally dispersed J-, H-, and K-band fringes
- Anamorphic cylindrical lens system and grism spectrograph



Examples of interferograms

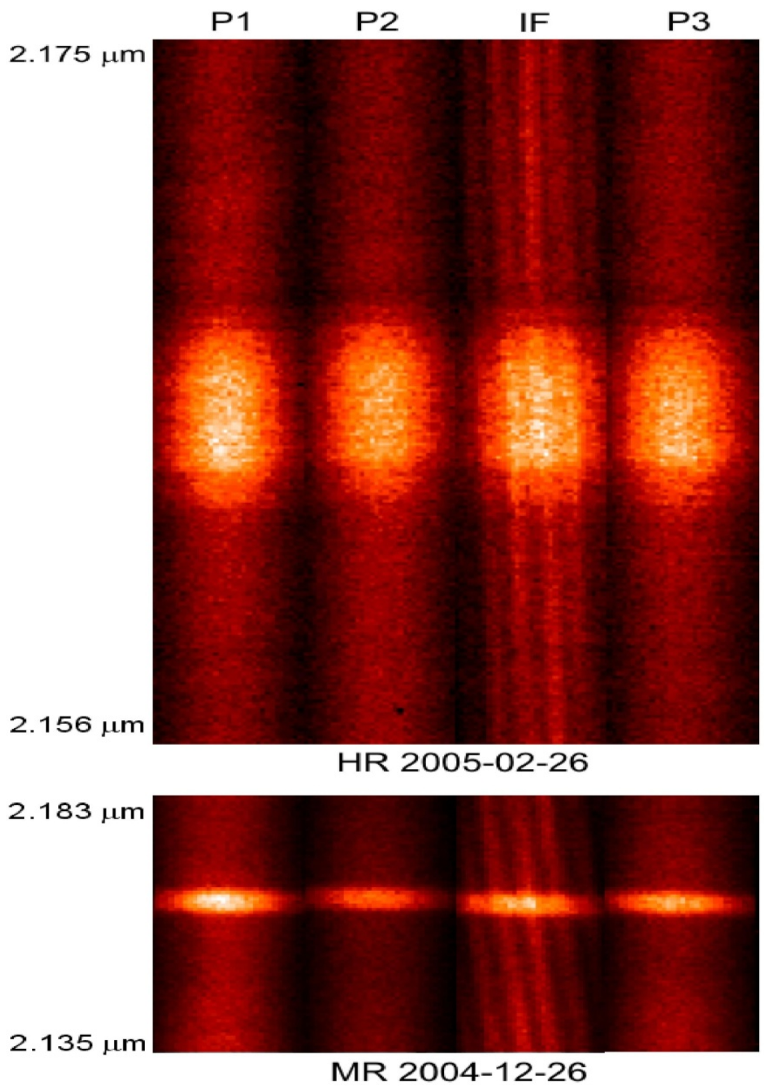


VLTI/ATs: interferograms of a bright star (low spectral resolution mode)



VLTI/UTs: NGC 3783; K magnitude = 10.1 (Weigelt et al. 2012)

Examples of interferograms



Eta Carinae
Br γ 2.166 µm emission line
Spectral resolution λ/Δλ = 12 000

Eta Carinae
Br γ 2.166 µm emission line
Spectral resolutions λ/Δλ = 1500

Weigelt et al. 2007, A& A 464, 87

Interferograms

Example: telescope + pupil mask with 2 small aperture holes (e.g. ~ 10 cm).

If the distance between the 2 holes is 4 m and the wavelength is 500 nm

→ fringe period λ/D ~ 30 mas

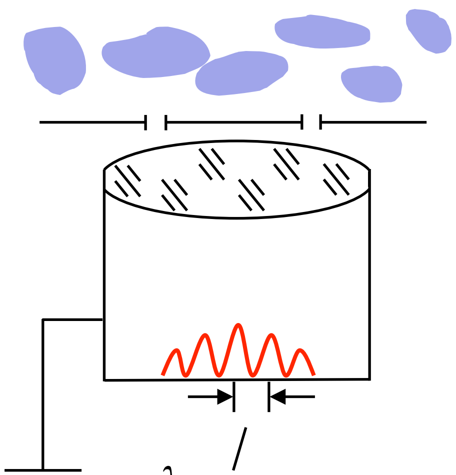
→ fringe contrast ~ 0 if the binary separation is ~ 15 mas (for equally bright components)

* single star

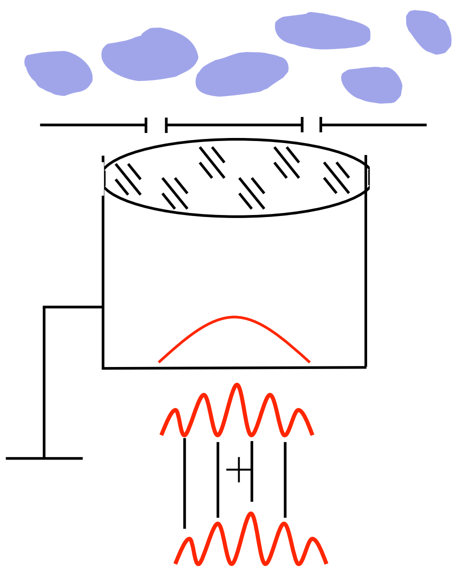
** binary star (equally bright components, separation = 0.015”

turbulence cells

→ fringe motion

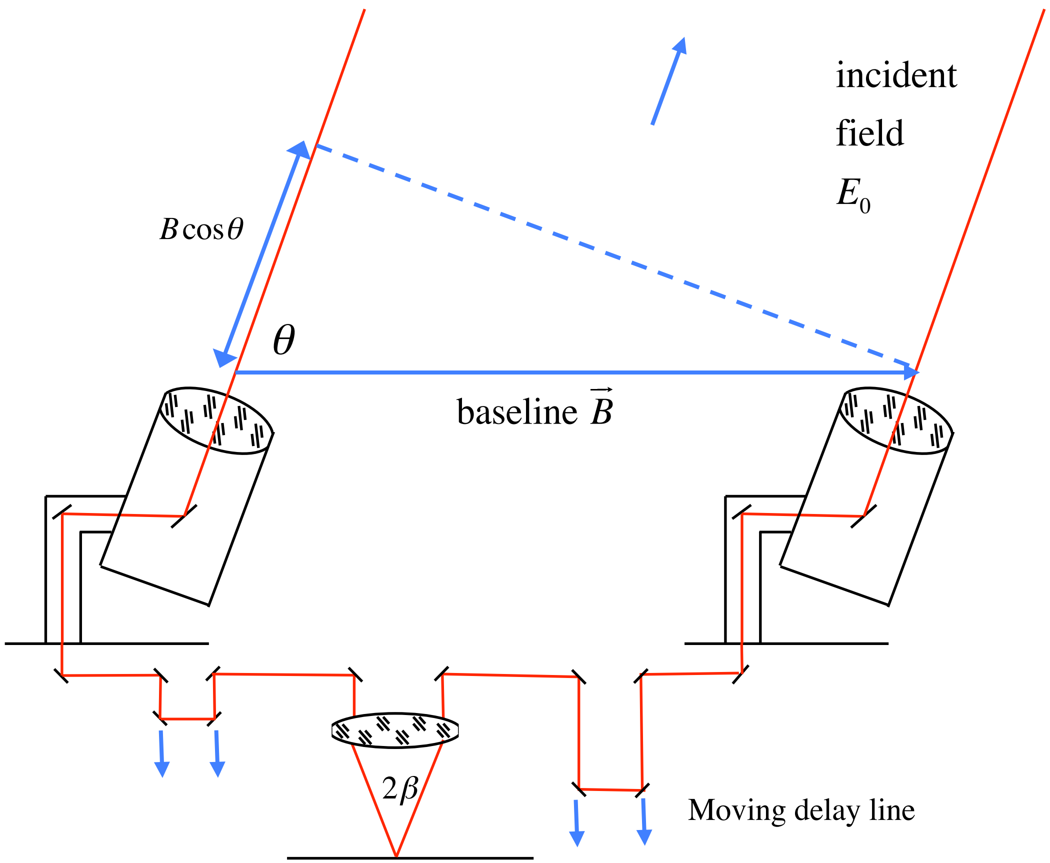


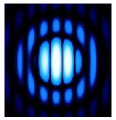
$$\frac{\lambda}{D} \approx 0.03'' \text{ when } D = 4 \text{ m}, \lambda = 550 \text{ nm}$$



2-telescope interferometer

intensity distribution of the object





intensity distribution $i(x) = ...?$ Object information in $i(x)$?

Interferograms are obtained if the pathlength difference is smaller than the

$$\text{coherence length} = \lambda\frac{\lambda}{\Delta\lambda}.$$

Example:

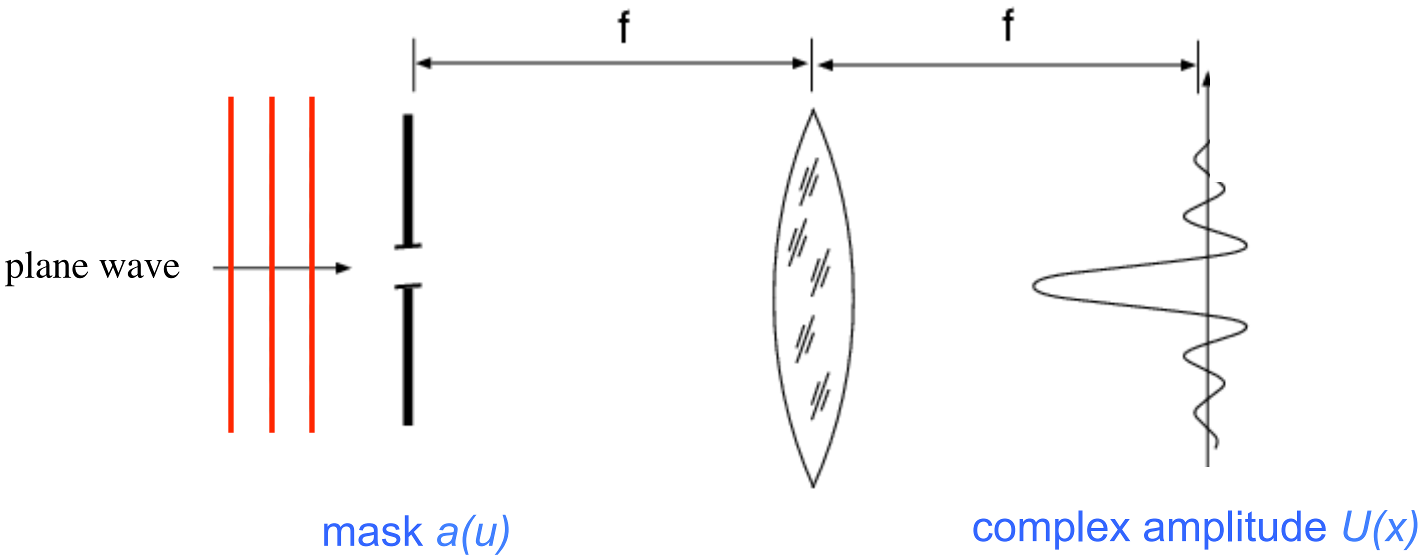
$\lambda = 1\ \mu\text{m}$,

filter width $\Delta\lambda = 0.001\ \mu\text{m}$

$\rightarrow$

coherence length $= 1$ mm

Fourier transform properties of lenses; complex amplitude



The electrical field E of a wave can be described by $E = \mathrm{Real}\left[\left|\Psi\right| \exp(i\Phi) \exp(-2\pi i \nu t) \right]$.

The low-frequncy part

$$\Psi = \left|\Psi\right| \exp(i\Phi)$$

is called the **complex amplitude** (or complex wave amplitude).

Fourier transformation properties of lenses

If $a(u,v)$ is the complex amplitude function in front of a lens,

then the **complex amplitude** in the focal plane of the lens is equal to the **Fourier transform**

$$U(x,y) = \text{phase factor} \int\int a(u,v)\, \exp\left[-\frac{2\pi i}{\lambda f}(ux+vy)\right] du\, dv = FT\left[a(u,v)\right] \rightarrow$$

– A function U can be constructed as the sum of many cos & sine functions of different frequencies; cos & sine because of the Euler theorem $\exp(iz) = \cos z + i \sin z$.

– Each individual exp function has a certain frequency and its own weight factor.

– The Fourier transform a of U gives the weight factors.

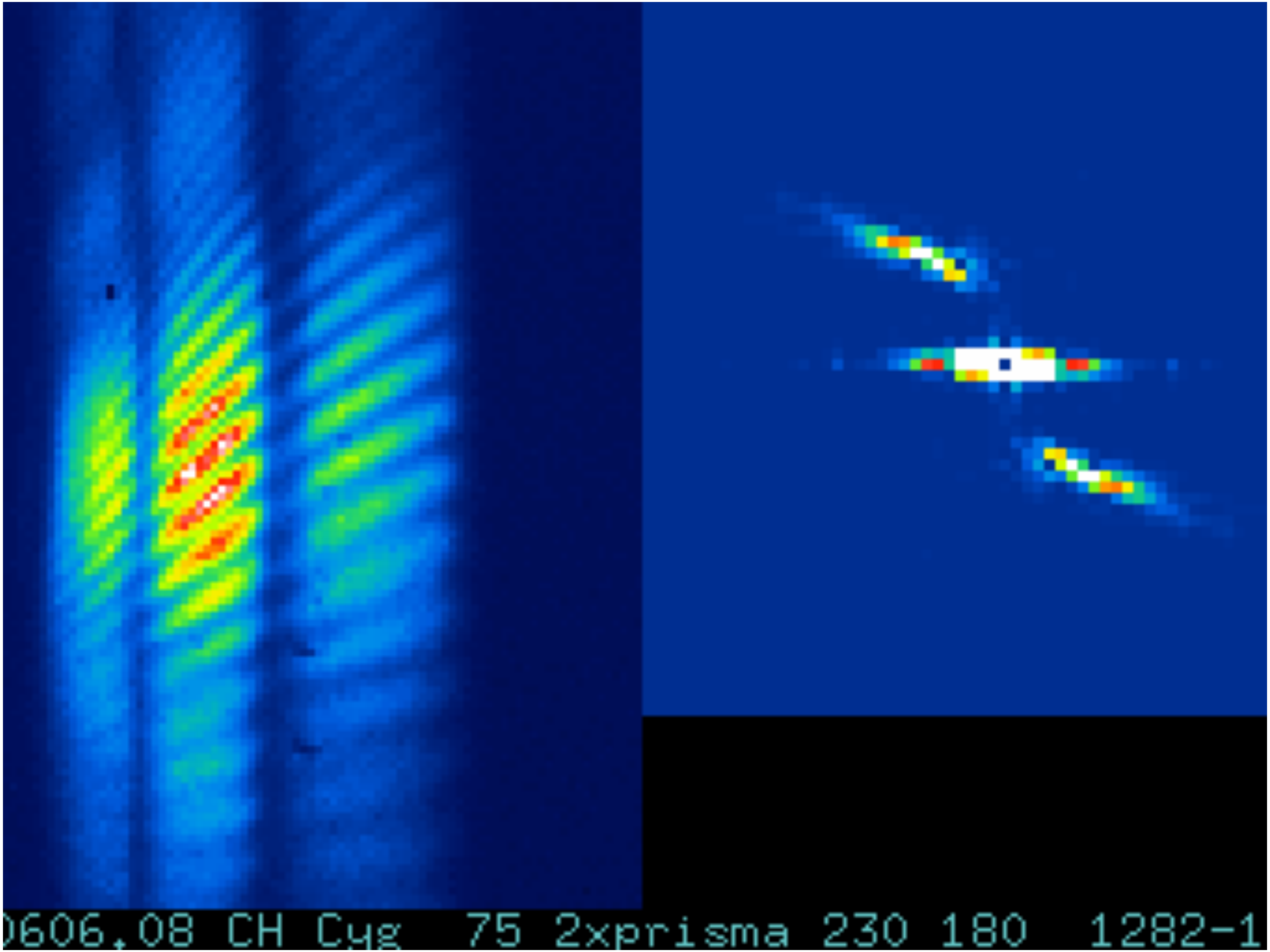
- One can also define $\exp[+...]$ instead of $\exp[-...]$ and all integrals are $\int_{-\infty}^{+\infty}$

Definition of the inverse Fourier transform $\mathrm{FT}^{-1}\left[U(x,y)\right]$:

$$a(u,v) = \text{phase factor} \int\int U(x,y) \exp\left[+\frac{2\pi i}{\lambda f}(ux+vy)\right] dxdy = \hat{F}^{-1}\left[U(x,y)\right]$$

See, e.g., J. W. Goodman book (1968): Introduction to to Fourier optics

Next slide : Real time Fourier transform movie (IOTA interferometer):
(left) JHK interferogram, (right) power spectum (modulus square of Fourier transform)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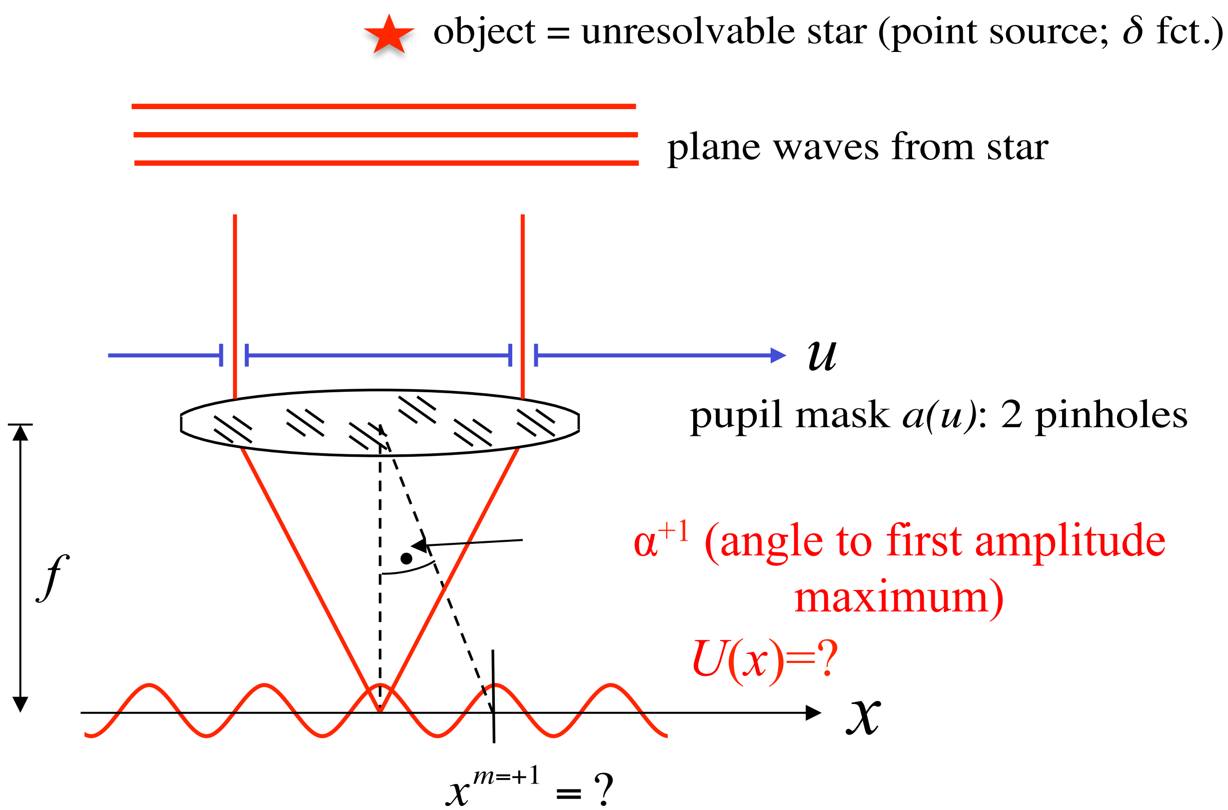
Interferograms: 2-hole mask, fringe separation, interferometer resolution (1)



Pupil function $a(u) = \left[\delta\left(u - u'\right) + \delta\left(u + u'\right)\right]$

Where is the first maximum (+1 maximum at $x^{m=+1}$) of the complex amplitude of $U(x)$? $x^{m=+1} = ?$

Interferograms: 2-hole mask, fringe separation, interferometer resolution (2)

Example : Point source object and pupil mask with amplitude transmission

$a(u) = \delta(u - u') + \delta(u + u')$ in front of a telescope.

Then we obtain the complex amplitude $a(u) = \delta(u - u') + \delta(u + u')$ in front of the lens.

The complex amplitude $U(x)$ in the focal plane of the telescope is equal to the Fourier transform of $a(u)$. $U(x)$ is also called Fraunhofer diffraction amplitude:

$$U(x) = \int \left[\delta(u - u') + \delta(u + u')\right] \exp\left(-\frac{2\pi i}{\lambda f} ux\right) du \quad \rightarrow *$$

$$U(x) = \exp\left(-\frac{2\pi i}{\lambda f} xu'\right) + \exp\left(+\frac{2\pi i}{\lambda f} xu'\right) = 2\cos\frac{2\pi}{\lambda f} xu' \quad \rightarrow$$

The focal plane intensity distribution $I(x) = \left|U(x)\right|^2 = 4\cos^2\frac{2\pi}{\lambda f} xu'$

---

* used definition of the Delta function was used: $\int f(x)\delta(x - a)\, dx = f(a)$; $\cos z = \frac{1}{2}\left(e^{zi} + e^{-zi}\right)$

Interferograms: 2-hole mask, fringe separation, interferometer resolution (3)

$\rightarrow$ Maxima of complex amplitude cosine for $\frac{1}{\lambda f} x u' = 0,\ 1,\ 2, \ldots$ $\rightarrow$

$\rightarrow +1$-maximum at $x = x^{m=1}$ obtained from $\frac{1}{\lambda f} x^{m=1} u' = 1$

$\rightarrow$ The $+1$-maximum of the complex amplitude lies at $x^{m=1} = \frac{\lambda f}{u'}$.

Let α^{+1} be the angle between the optical axis and the line to the $+1$-amplitude maximum.

Then, $\alpha^{+1} = \frac{x^{m=1}}{f} = \frac{\lambda}{u'}$ (for small angles and because $x^{m=1} = \frac{\lambda f}{u'}$).

Then, the $+1$-intensity maximum lies at $\alpha^{+1\ \text{intensity}} = \frac{1}{2} \frac{\lambda}{u'} = \frac{\lambda}{B}$

If $B = 2u'$ is the baseline length (= separation of the 2 holes)

Therefore, an interferometer with baseline length B is able to resolve objects as small as $\frac{\lambda}{B}$.

Example: λ=2.2 μm, B=100 m $\rightarrow \frac{\lambda}{B} = \frac{2.2 \cdot 10^{-6}\,\text{m}}{100\,\text{m}}$ corresponding to $\frac{2.2 \cdot 10^{-6}\,\text{m}}{100\,\text{m} \cdot 4.8 \cdot 10^{-6}} = 4.6$ mas

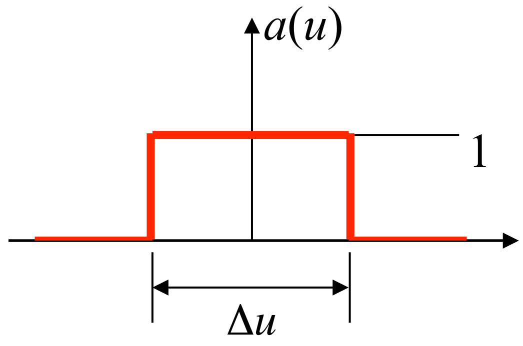
Fourier transform of a rectangular (rect) function *a(u) with width Δu*

$$U(x) \propto \int_{-\infty}^{+\infty} a(u) \exp\left(-\frac{2\pi i}{\lambda f} ux\right) du$$

$$= \int_{-\Delta u/2}^{+\Delta u/2} 1 \exp\left(-\frac{2\pi i}{\lambda f} ux\right) du =* \left[\frac{\exp(-2\pi iux/\lambda f)}{-2\pi ix/\lambda f}\right]_{-\Delta u/2}^{+\Delta u/2}$$

$$= -\frac{\lambda f}{\pi x} \frac{\exp(-\pi ix\Delta u/\lambda f) - \exp(+\pi ix\Delta u/\lambda f)}{2i}$$

$$= -\frac{\lambda f}{\pi x} \sin\frac{-\pi x\Delta u}{\lambda f} = \frac{\lambda f}{\pi x} \sin\frac{\pi x\Delta u}{\lambda f}$$



Because $\sin z = \dfrac{e^{zi} - e^{-zi}}{2i}$, $\left(\exp\left(-\dfrac{2\pi iux}{\lambda f}\right)\right)' = -2\pi ix \exp\left(-\dfrac{2\pi iux}{\lambda f}\right)$

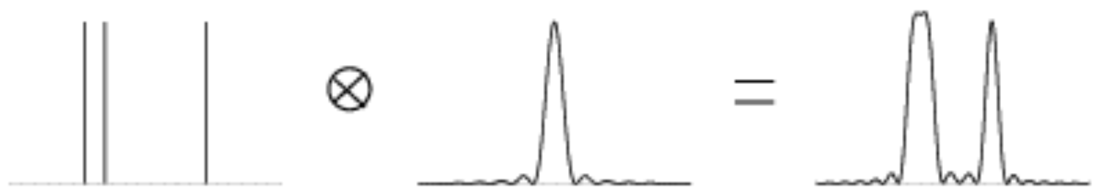
Convolution and convolution theorem

$i(x) = \int g(\xi)h(x-\xi)d\xi = g(x)\otimes h(x)$ (= g(x) ⊗ h(x); ⊗ = convolution operator)

= convolution of g(x) with h(x)

Convolution theorem: The Fourier transform of a convolution of 2 functions
is equal to the product of the Fourier transforms of the 2 functions (derivation in the Appendix)

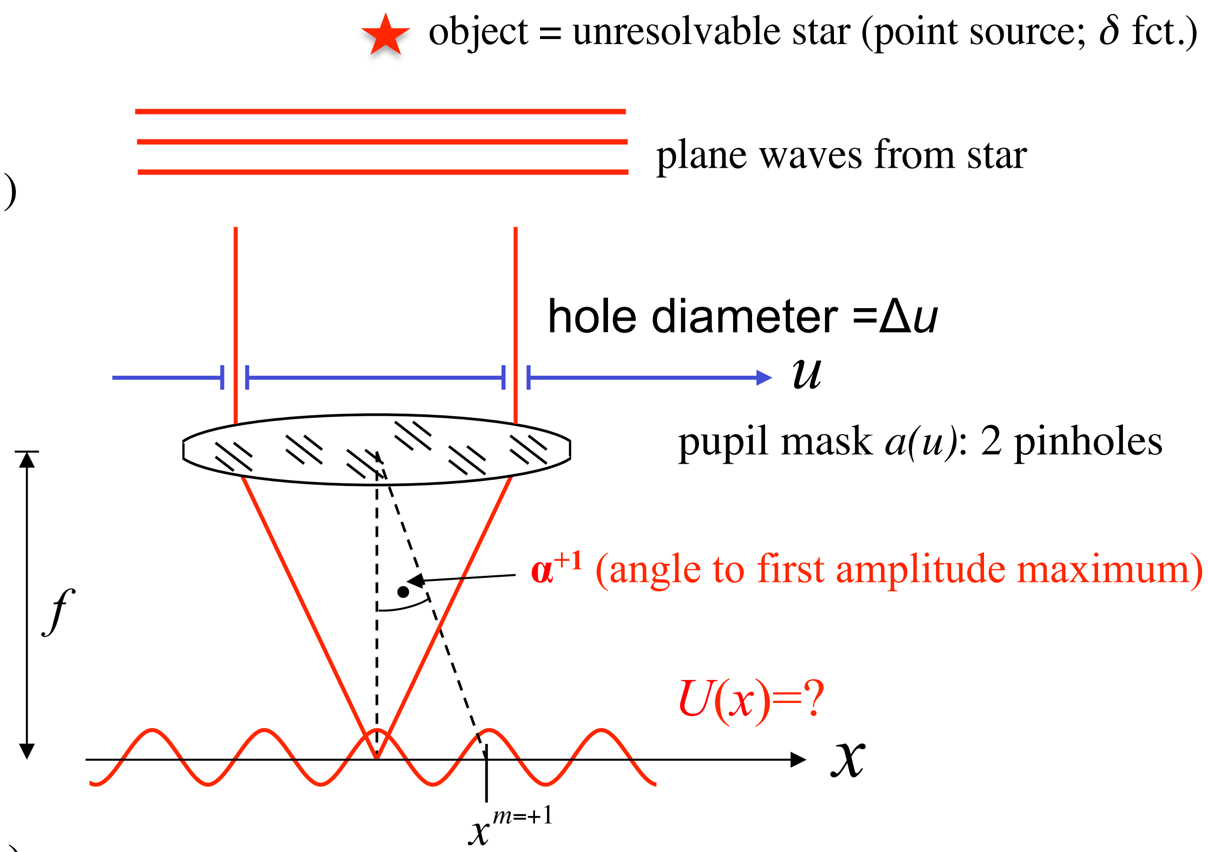
1D convolution example:

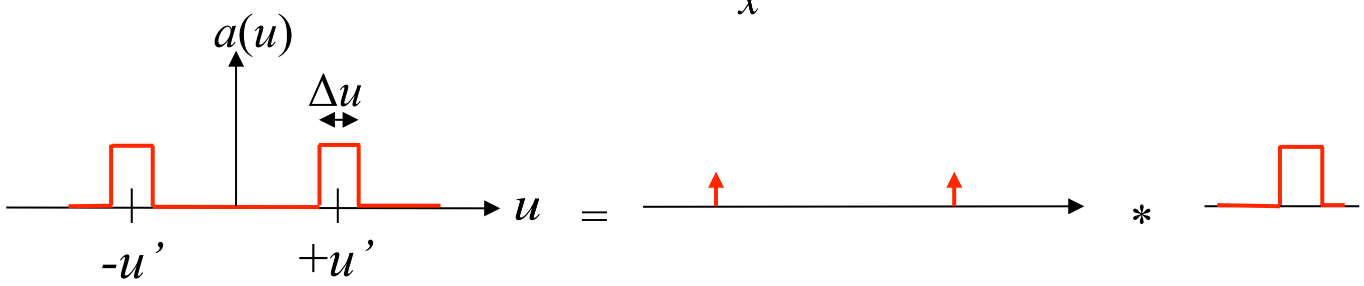


Interferograms: 2-hole mask, fringe separation, interferometer resolution (5)
2-hole pupil mask with extended holes

Where is the first maximum (+1 maximum at $x^{m=+1}$) of the complex amplitude of $U(x)$?

$x^{m=+1} = ?$



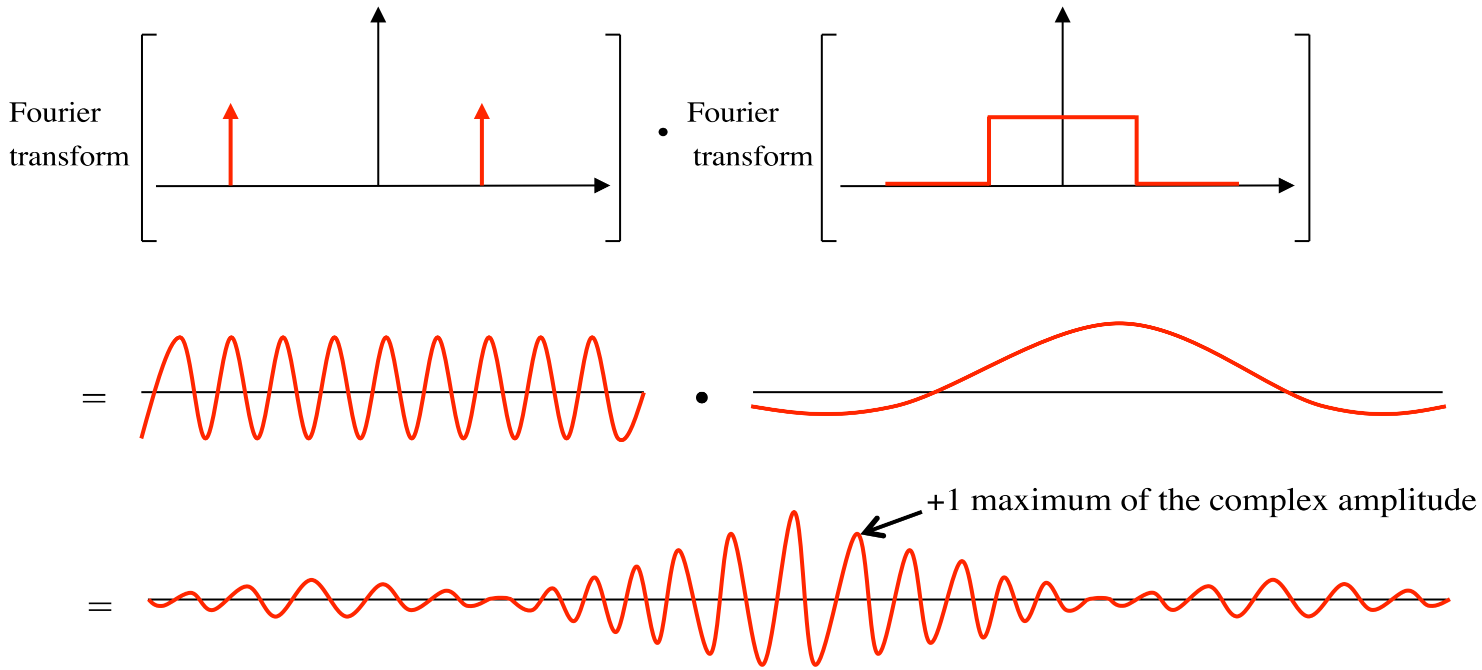


Interferograms: 2-hole mask, fringe separation, interferometer resolution (6)
2-hole pupil mask with extended holes

$$\rightarrow U(x) = \int \left[\delta(u-u') + \delta(u+u')\right] \exp\left[-2\pi i u x\right] du \cdot \int rect\frac{u}{\Delta u} \exp\left[-2\pi i u x\right] du \rightarrow$$

$$U(x) = 2\cos\frac{2\pi}{\lambda f} x u' \cdot \frac{\lambda f}{\pi x} \sin\frac{\pi x \Delta u}{\lambda f}$$

(* because of convolution theorem and previous 2 examples). Illustration of all 5 functions:



A few important facts from wave or Fourier optics

(see, e.g., J. W. Goodman book (1968): Introduction to to Fourier optics):

(1) Convolution and convolution theorem :

Convolution of $o(x)$ with $p(x) \equiv \int o(x')p(x-x')dx' = o(x) \otimes p(x)$ ($\otimes$ = convolution operator)

Convolution theorem: The Fourier transform of a convolution of 2 functions is equal to the product of the Fourier transforms of these 2 functions (see derivation in the Appendix)

(2) Incoherent, space - invariant imaging equation :

The image intensity distribution $i(x)$ is equal to the convolution of the object intensity distribution $o(x)$ with the intensity distribution of the point spread function $p(x)$, if $p(x)$ is space-invariant: $i(x) = o(x) \otimes p(x)$

(3) Fourier transform (FT) property of a lens and point spread function p(x) :

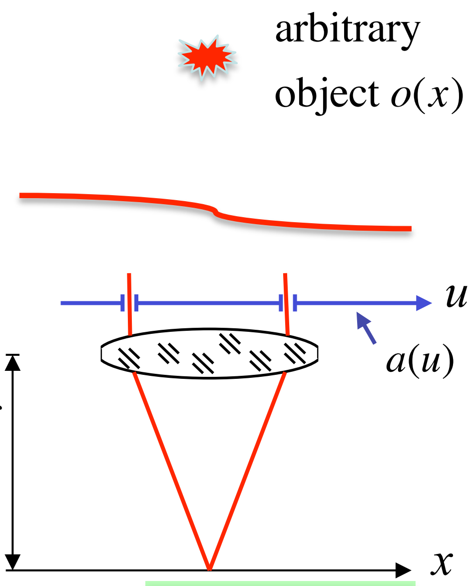
The intensity distribution $p(x)$ of the image of a point source is equal $p(x) = \left|FT\left[\text{pupil function } a(u)\right]\right|^2$.

(4) Autocorrelation & autocorrelation theorem :

The autocorrelation $AC\left[u\right]$ of $a(u)$ is equal to $AC\left[u\right] \equiv \int a(u'+u)a^*(u')du'$

Autocorrelation theorem: $FT^{-1}\left[\left|FT\left(a(u)\right)\right|^2\right] =$ autocorrelation $AC\left[a(u)\right]$ (see derivation in the Appendix)

Interferogram *i(x)* of an arbitrary object *o(x)* (1)



Calculation of the intensity distribution $i(x)$ of the interferogram, if the object is an arbitrary object with intensity distribution $o(x)$ (in contrast to the point source object in the previous calculations).

The pupil function $a(u)$ is assumed to consist of 2 pinholes, i.e., 2 delta functions: $a(u) = \left[\delta(u-u_1)+\delta(u-u_2)\right]$

Intensity distribution $i(x)$ of the interferograms is:

$i(x) = ?$
Dependence on $o(x)$?

$$i(x) = o(x) \otimes \left|FT\left[a(u)\right]\right|^2 \rightarrow I(u) = O(u)FT^{-1}\left[\left|FT\left[a(u)\right]\right|^2\right] = O(u)AC\left[a(u)\right]$$

(if $I(u)$ & $O(u)$ are the Fourier transforms of $i(x)$ & $o(x)$, respectively)

$$\rightarrow I(u) = O(u)\left\{2\delta(u-0)+\delta\left[u-(u_2-u_1)\right]+\delta\left[u-(u_1-u_2)\right]\right\}$$

Because of convolution and AC theorem; $AC\left[a(u)\right] = \int\left[\delta(w-u_1+u)+\delta(w-u_2+u)\right]\cdot\left[\delta(w-u_1)+\delta(w-u_2)\right]dw$

$$= \int\left[\delta(w-u_1+u)\right]\cdot\left[\delta(w-u_1)\right]dw + \ldots = 2\delta(u-0)+\delta\left[u-(u_2-u_1)\right]+\delta\left[u-(u_1-u_2)\right]$$

(with, e.g., substitution $w' = w-u_1+u$ etc.)

Interferogram *i(x)* of an arbitrary object o(x) (2)

$$\rightarrow I(u) = O(0)2\delta(u-0) + O(u_2-u_1)\delta[u-(u_2-u_1)] + O(u_1-u_2)\delta[u-(u_1-u_2)]$$

$$* \rightarrow i(x) = 2O(0) + O(u_2-u_1)\exp\left[2\pi i\frac{1}{\lambda f}(u_2-u_1)x\right] + O(u_1-u_2)\exp\left[2\pi i\frac{1}{\lambda f}(u_1-u_2)x\right]$$

$$** \rightarrow i(x) = 2O(0) + O(u_2-u_1)\exp\left[2\pi i\frac{1}{\lambda f}(u_2-u_1)x\right] + O^*(u_2-u_1)\exp\left[-2\pi i\frac{1}{\lambda f}(u_2-u_1)x\right]$$

= 2O(0) + c + c*

Now $i(x)$ has the form $i = 2O(0) + c + c^*$.

---

* We used: $o(x) =$ real $\rightarrow O(u) = O^*(-u)$

we used: $\int f(x)\delta(x-a)\,dx = f(a) \rightarrow \int \delta[u-(u_2-u_1)]\exp[2\pi ibux]du = \exp[2\pi ib(u_2-u_1)x]$

** since: we want to get the form $c + c^*$ (c = complex number),

which allows us to use $c + c^* = 2\ \mathrm{Re}(c)$:

$a + ib + a - ib = 2a = 2\ \mathrm{Re}(a+ib)$; if $o =$ real $\rightarrow O(u_1-u_2) = O^*(u_2-u_1)$)

Interferogram *i(x)* of an arbitrary object *o(x)* (2)

$c_1 \quad \times \quad c_2$

$$* \rightarrow i(x) = 2O(0) + 2\mathrm{Re}\left\{ O(u_2 - u_1) \cdot \exp\left[2\pi i \frac{1}{\lambda f}(u_2 - u_1)x \right] \right\}$$

$$\rightarrow \quad i(x) = 2O(0) + 2\left| O(u_2 - u_1) \right| \cos\left[2\pi \frac{1}{\lambda f}(u_2 - u_1)x + \arg\left[O(u_2 - u_1) \right] \right].$$

The interferogram $i(x)$, therefore, contains information on both

(1) the modulus $\left| O(u_2 - u_1) \right|$ and

(2) the phase $\arg\left[O(u_2 - u_1) \right]$ of the object Fourier transform at the baseline $u_2 - u_1$.

However, the atmospheric phase differences in front of both mask openings lead to additional statistical displacements of the interferogram (next slide: unwanted & unknown phase in the cos).

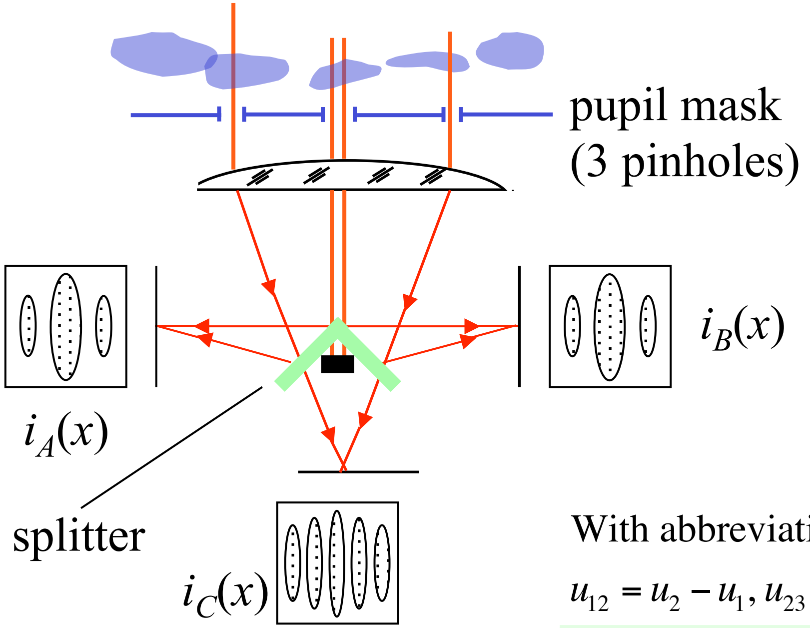
---

* The above equation has the form $i = 2O(0) + 2 \text{ Real } \{c_1 c_2\}$.

We use: $\text{Real } \{c_1 c_2\} = \text{ Real}\left\{ |c_1| \exp\left[i \arg(c_1) \right] |c_2| \exp\left[i \arg(c_2) \right] \right\}$

$= |c_1| |c_2| \cos\left(\arg(c_1) + \arg(c_2) \right)$ (because of $\exp(iz) = \cos z + i \sin z$))

Phase closure method (1) (Jennison, MNRAS 118, 276, 1958)



With abbreviations:

$u_{12} = u_2 - u_1, u_{23} = u_3 - u_2,\ u_{13} = u_3 - u_1$, and, for example,

$\arg\left[O(u_{12})\right] = \phi(u_{12}) =$ **phase** of the object Fourier transform at u_{12}.

$|O(u_{12})| =$ **modulus** of the object Fourier transform at baseline u_{12}.

Additionally: $\varphi_1,\ \varphi_2,\ \&\ \varphi_3$ = unknown atmospheric phases in front of the mask holes! → phase problem!

$$\rightarrow i_A(x) \propto 1 + |O(u_{12})| \cos\left[2\pi \frac{1}{\lambda f} u_{12} x + \phi(u_{12}) + \varphi_1 - \varphi_2\right],$$

(see shift theorem in the Appendix)

$$i_B(x) \propto 1 + |O(u_{23})| \cos\left[2\pi \frac{1}{\lambda f} u_{23} x + \phi(u_{23}) + \varphi_2 - \varphi_3\right],\quad i_C(x) \propto 1 + |O(u_{13})| \cos\left[2\pi \frac{1}{\lambda f} u_{13} x + \phi(u_{13}) + \varphi_1 - \varphi_3\right]$$

Phase closure method (2)

Example with baselines as in the previous slide: Evaluation of $i_A, i_B,$ and i_C (with $b := u_{12} = u_{23}, 2b = u_{13}$):

phase of i_A is called $\theta_A(b) = \phi(b) + \varphi_1 - \varphi_2$; θ phases are called "**dirty**" **phases**"

phase of i_B is called $\theta_B(b) = \phi(b) + \varphi_2 - \varphi_3$, phase of i_C is called $\theta_C(2b) = \phi(2b) + \varphi_1 - \varphi_3$

θ_C, θ_A, & θ_B are called the measured "dirty phases", ϕ is the want object Fourier phase, and φ_1, φ_2, & φ_3 are the unknown atmospheric phases. We calculate the following **phase difference, called closure phase**:

$$\text{closure phase} \equiv \theta_C - \theta_A - \theta_B = \phi(2b) + \varphi_1 - \varphi_3 - \phi(b) - \varphi_1 + \varphi_2 - \phi(b) - \varphi_2 + \varphi_3 = \phi(2b) - \phi(b) - \phi(b)$$

$\rightarrow \quad \phi(2b) = 2\phi(b) + \theta_C(2b) - \theta_A(b) - \theta_B(b)$; analog $\phi(3b) = \ldots, \phi(4b) = \ldots$, etc.;

Atmospheric phases cancelled!

However, in real observations, one cannot measure at such regular distance steps.

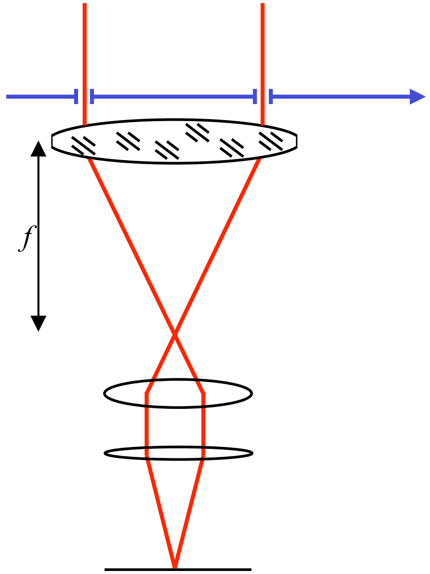
For interferograms θ_C, θ_A, and θ_B with 3 arbitrary baseline vectors $\vec{u}_3$, $\vec{u}_1$, and $\vec{u}_2$ that form a closed triangle (instead of the above 3 special baselines):

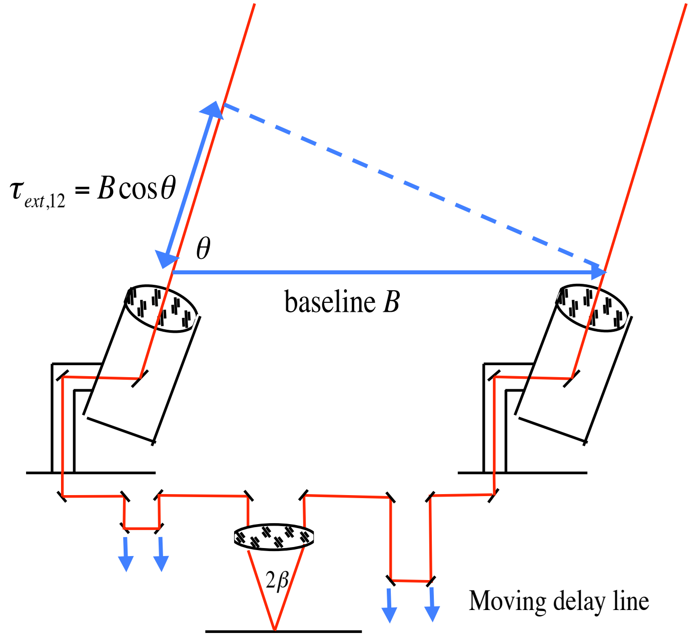
Object Fourier phases

$$\textbf{closure phase} \equiv \theta_{13} - \theta_{12} - \theta_{23} = \phi(\vec{u}_{13}) + \varphi_1 - \varphi_3 - \phi(\vec{u}_{12}) - \varphi_1 + \varphi_2 - \phi(\vec{u}_{23}) - \varphi_2 + \varphi_3 = \phi(\vec{u}_3) - \phi(\vec{u}_1) - \phi(\vec{u}_2).$$

From many such closure phase measurements $\theta_{13} - \theta_{12} - \theta_{23}$ made with many different triangle configurations, an image of the object can be reconstructed (talk by Karl-Heinz Hofmann).

Comparison of interferometers with two different pupil arrangements





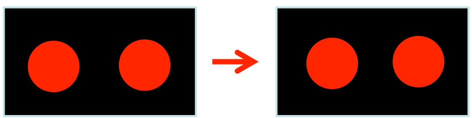
1. **Exit pupil is scaled - down version of entrance pupil**:

Advantage: space-invariant psf & large FOV (e.g., 30")

Disadvantage: low resolution

Example: telescope with a pupil mask, LBTI * (2x8 m)

Entrance pupil: exit pupil:



2. **Exit pupil is not a scaled - down version of input pupil.**

Advantage: smaller number of pixels per fringe

Disadvantage: small FOV (e.g., 70 mas)

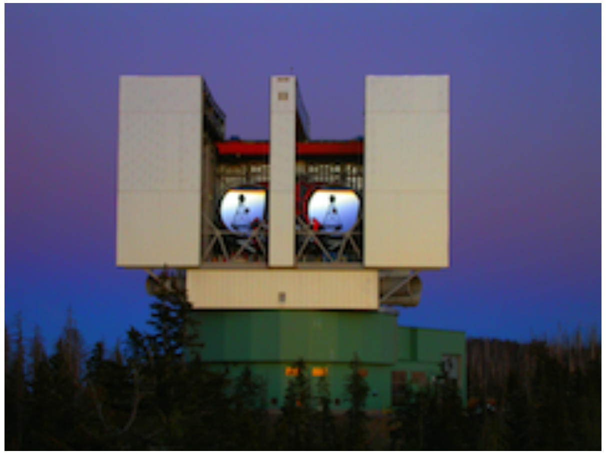
Examples: VLTI, CHARA, NPOI etc.

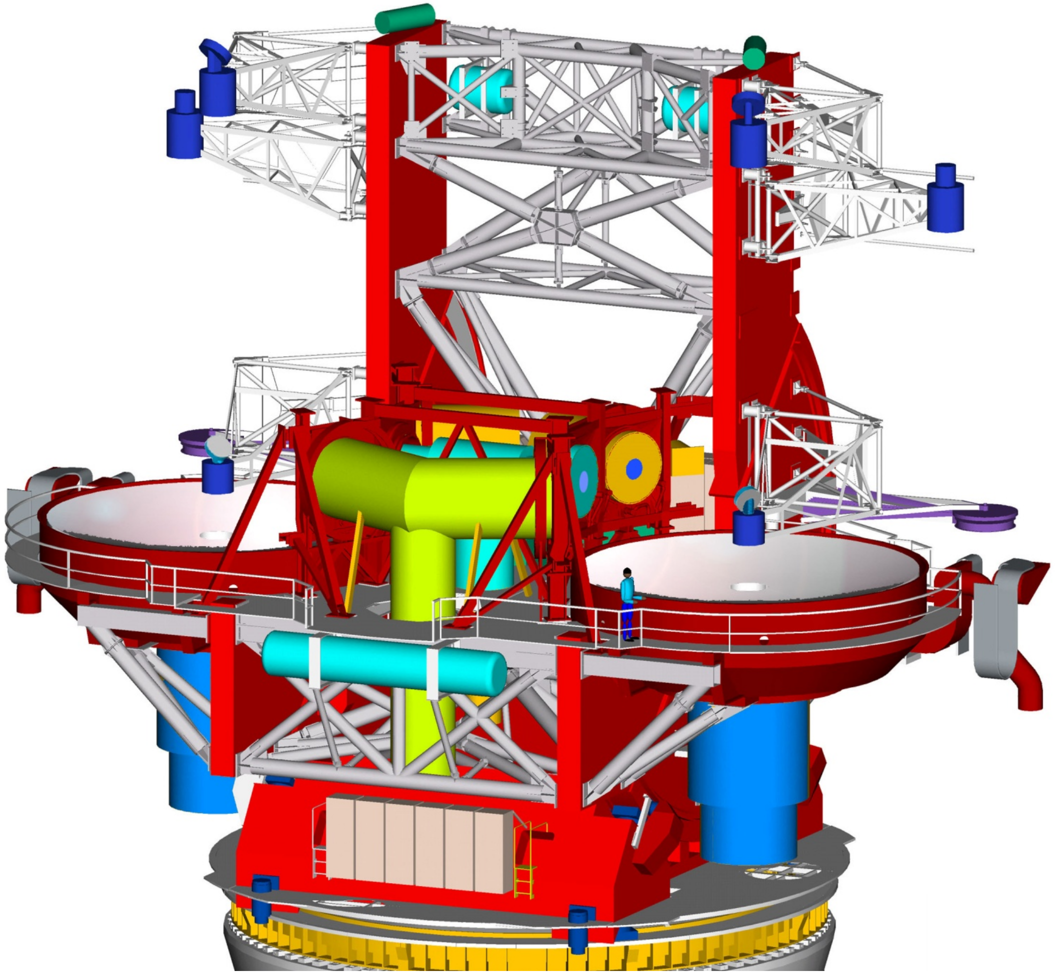
Entrance pupil exit pupil

The following calculation of $i(x)$ assumes the above pupil reconfiguration (right)

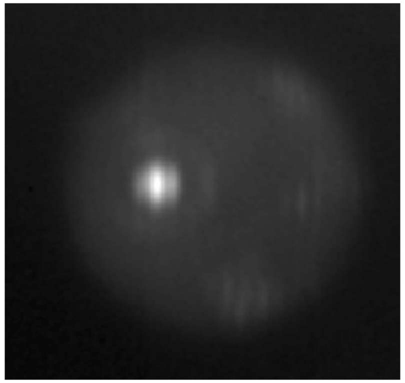
* Next slide: LBTI example

LBT

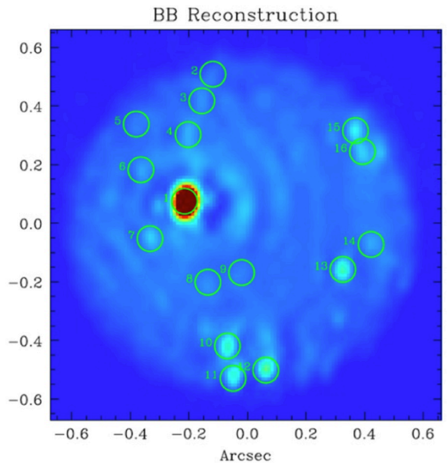




First LBTI observations: Image of 16 volcanos on of Jupiter‘s moon of Io



One of the recorded interferograms



Reconstructed image

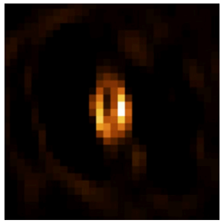
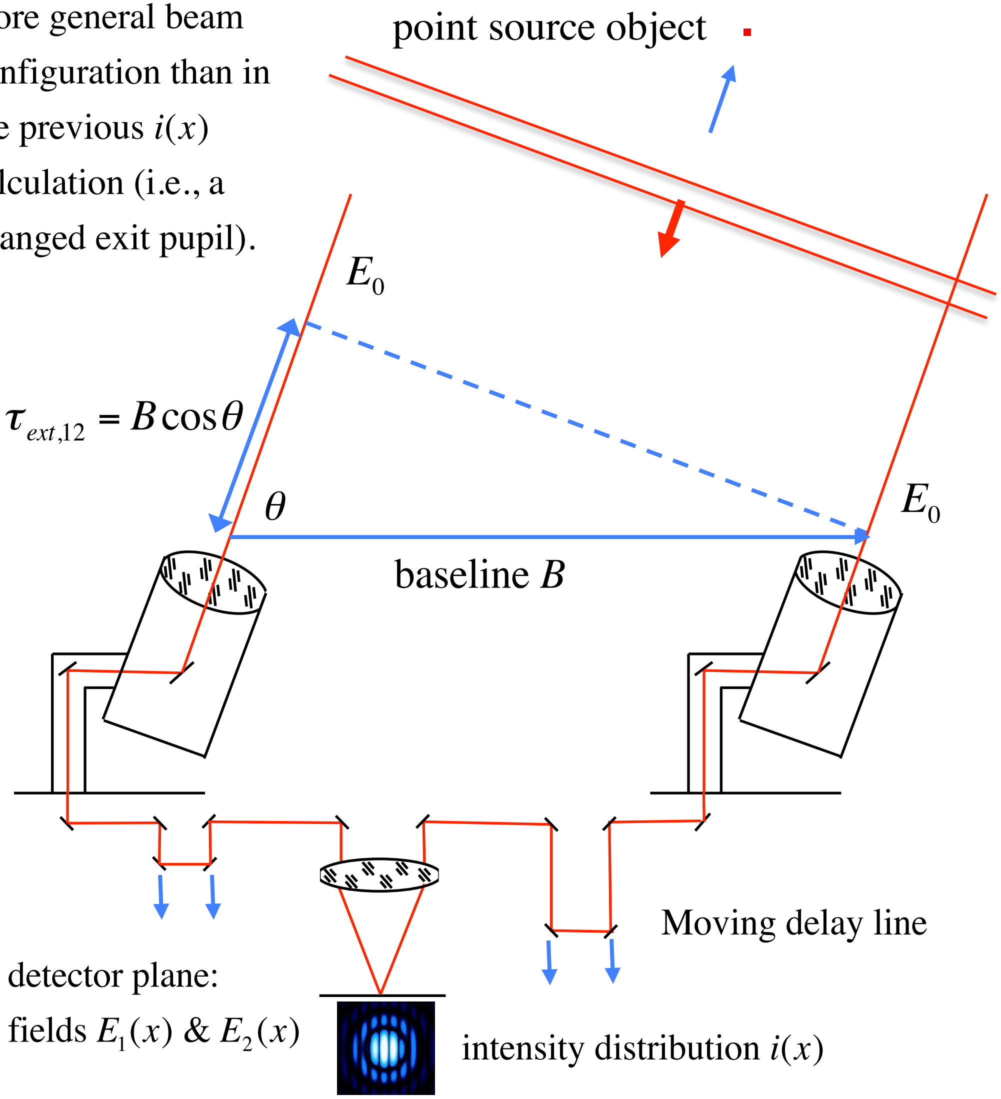


Image of the brightest volcano Loki with its Lava sea

Conrad et al. 2015, AJ 149, 175

2-telescope interferometer*: interferogram of an arbitrary object

* Now we discuss a more general beam configuration than in the previous $i(x)$ calculation (i.e., a changed exit pupil).



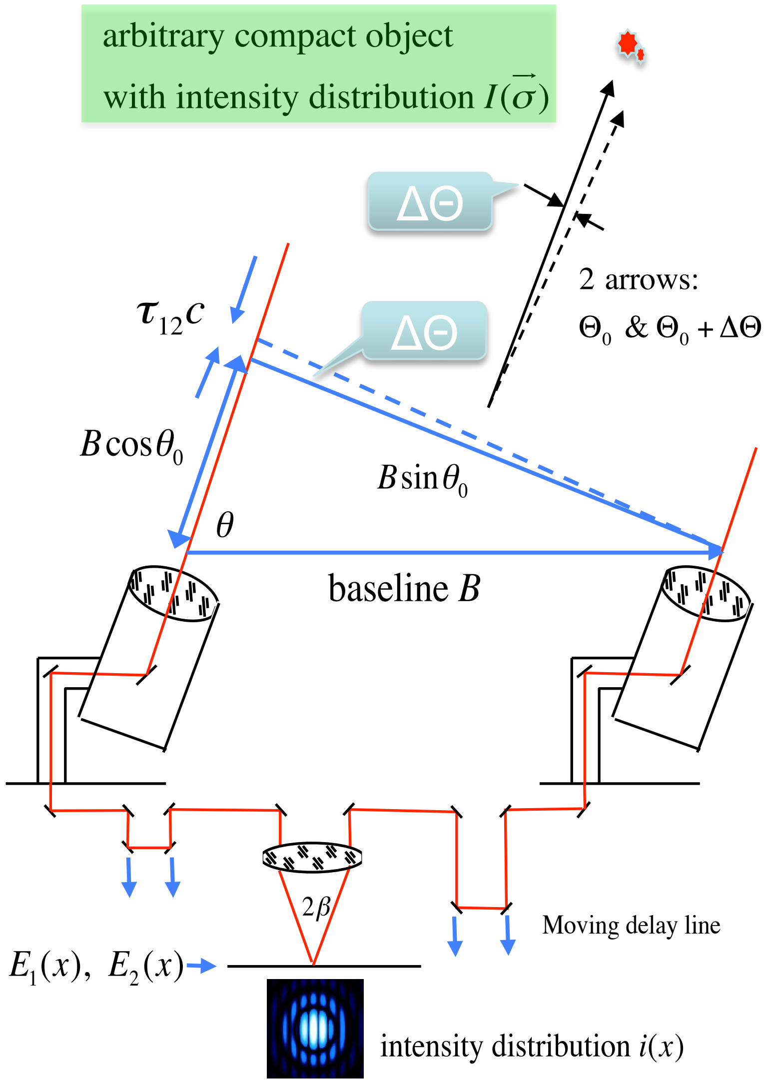
Useful references for the following calculations:
Goodman books on Fourier and statistical optics; Thompson et al. 1986; Boden 1999; Haniff 1999; the following slides present the theory reported in the book "Practical Optical Interferometry" (Buscher 2015) plus some additional calculations.

The incident monochromatic field E_0 (red plane waves) can be described (except constants) by

$$E_0 = \text{Re}\left[|\Psi_0|\exp(i\Phi_0)\exp(-2\pi i\nu t)\right]$$

The low-frequency part
$\Psi_0 = |\Psi_0|\exp(i\Phi_0)$
is called the complex amplitude.
The mean intensity or flux is the time average $F = \langle E^2 \rangle$

The astrometric phase: optical path difference of a nearby source



The figure shows that external time delay difference between the 2 beams of telescope 1 and 2 is, for a point source (1 star),

$$\tau_{ext,12} = \tau_{ext,1} - \tau_{ext,2} = B\frac{\cos\theta_0}{c}$$

The delay line system can compensate this external delay for one point source to get $\tau_{ext,12} = 0$.

However, the optical path **difference** $\tau_{12}c$ between a point source in direction of θ_0 and a second nearby point source in direction of $\theta_0 + \Delta\theta$ observed simultaneously, is (for small $\Delta\theta$)

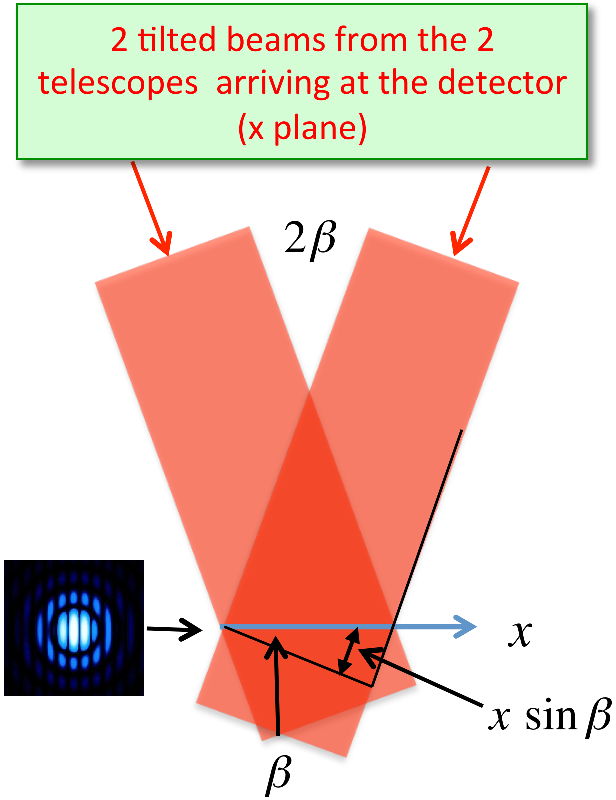
$$\tau_{12}c = B\cos(\theta_0 + \Delta\theta) - B\cos\theta_0 \approx -B\Delta\theta\sin\theta_0$$

Therefore, the phase shift ϕ_{12} (= 2π OPD/λ; 1 λ OPD corresponds to a phase shift of 2π; $\nu = c/\lambda$) of the fringes is

$$\phi_{12} = -2\pi\nu\tau_{12} = -2\pi\frac{1}{\lambda}B\sin\theta_0\Delta\theta = -2\pi\left(B\frac{\sin\theta_0}{\lambda}\right)\Delta\theta = -2\pi u\Delta\theta,$$

where $u = B\dfrac{\sin\theta_0}{\lambda}$ and u is the length of the projected baseline (as seen from the star). The phase shift of an interferogram of an offaxis object element depends on u and $\Delta\theta$.

The beam combiner (BC) time delay differences



We assume that a socalled multi-axial beam combination is used as in AMBER and MATISSE (and the previous IOTA movie).

In this case, the 2 telescope beams arriving at the detector are tilted against each other by an angle of 2β.

We have to analyze the beam combiner time delay difference $\tau_{BC,1}(x) - \tau_{BC,2}(x)$ caused because of the tilt 2β.

Rays arriving at position x of the detector have to travel an additional distance of $\pm x \sin\beta$ compared to the rays arriving at $x = 0$.

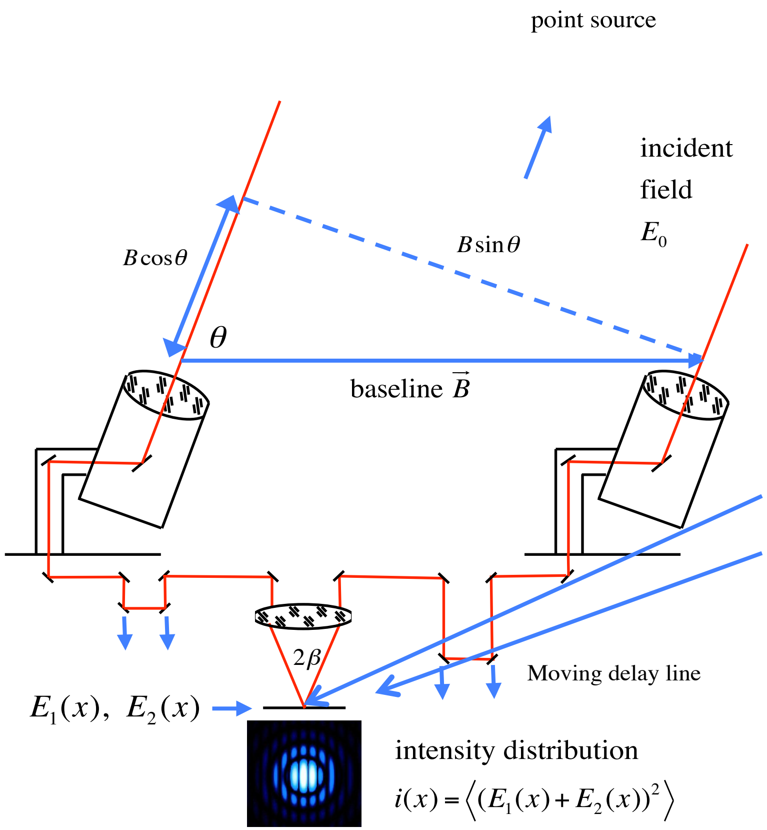
Then the optical path difference OPD_{BC} (defined as delay τ multiplied with the speed of light c; $\nu\lambda$=c) between the 2 beams varies with x as

$$\mathrm{OPD}_{BC} = c(\tau_{BC,1}(x) - \tau_{BC,2}(x)) = 2x\sin\beta$$

$$\rightarrow \nu\left(\tau_{BC,1}(x) - \tau_{BC,2}(x)\right) = x\frac{1}{c}\nu 2\sin\beta;\ \text{with abbreviation}\ \frac{1}{c}\nu 2\sin\beta \equiv s\ \rightarrow$$

$$\nu\left(\tau_{BC,1}(x) - \tau_{BC,2}(x)\right) = s\,x\,;\ s = \text{fringe frequeny}$$

Intensity distribution *i(x)* of an interferogram (1)



We still have to calculate these $\tau_{1,2}$ and $\tau_{BC,1}(x) - \tau_{BC,2}(x)$ →

Two waves E_1 and E_2 (frequency ν) arrive at the detector and interfere, and we observe the

interferogram intensity distribution $i(x) = \left\langle \left(E_1(x) + E_2(x)\right)^2 \right\rangle \rightarrow$

$$i(x) = \left\langle \left(\mathrm{Re}\left[\Psi_1(x)\exp(-2\pi i t\nu) + \Psi_2(x)\exp(-2\pi i\nu t)\right]\right)^2 \right\rangle \rightarrow$$

$$i(x) = \left|\Psi_1(x)\right|^2 + \left|\Psi_2(x)\right|^2 + 2\,\mathrm{Re}\left[\Psi_1(x)\Psi_2^*(x)\right] \quad \text{(proof on next slide)}$$

The waves arriving at the focal plane are time-delayed versions of E_0. Therefore, at the detector, waves with the following two complex amplitudes will arrive

$$\Psi_1 = \Psi_0 \exp\left[2\pi\nu(\tau_{ext,1} + \tau_{\mathrm{int},1} + \tau_{BC,1}(x)\right],$$

$$\Psi_2 = \Psi_0 \exp\left[2\pi\nu(\tau_{ext,2} + \tau_{\mathrm{int},2} + \tau_{BC,2}(x)\right] \rightarrow$$

$$\Psi_1\Psi_2^* = \left|\Psi_0\right|^2 \exp\left[2\pi\nu\left(\tau_{ext,1} - \tau_{ext,2} + \tau_{\mathrm{int},1} - \tau_{\mathrm{int},2} + \tau_{BC,1}(x) - \tau_{BC,2}(x)\right)\right]$$

The τ terms are external, internal & BC time delays for tel. 1 & 2.

If we insert this $\Psi_1\Psi_2^*$ into the above $i(x)$ equation and define

$\tau_{1,2} = \tau_{ext,1} - \tau_{ext,2} + \tau_{\mathrm{int},1} - \tau_{\mathrm{int},2}$ & $\left|\Psi_0\right|^2$ = mean intensity F_0, we obtain

$$i(x) = 2F_0\left(1 + \mathrm{Re}\left[\exp\left\{2\pi i\nu\left(\tau_{1,2} + \tau_{BC,1}(x) - \tau_{BC,2}(x)\right)\right\}\right]\right)$$

$\tau_{1,2}$ can be made to zero for a point source in a particular direction if the delay line sytem is adjusted to compensate the external delay.

Calculation of time average

$$i(x) = \left\langle \left(E_1(x)+E_2(x)\right)^2 \right\rangle = \left\langle E_1^2(x)+E_2^2(x)+2E_1(x)E_2(x) \right\rangle = \left\langle E_1^2(x)\right\rangle + \left\langle E_2^2(x)\right\rangle + \left\langle 2E_1(x)E_2(x)\right\rangle$$

Using: $\mathrm{Re}[c] = 0.5(c+c^*)$; $\mathrm{Re}[ab] = 0.5\left(ab+a^*b^*\right)$; use abbreviation: $b_i \equiv b_1 \equiv b_2 \equiv \exp(-2\pi i\nu t)$; index $i = 1$ or 2

$E_i(x) = \mathrm{Re}\left[\psi_i \exp(-2\pi i\nu t)\right] = ?$ ($\psi_i \cdot \exp(-2\pi i\nu t)$ is a product of 2 complex numbers);

$\mathrm{Re}[ab] = 0.5\left(ab+a^*b^*\right) \to \mathrm{Re}\left[\psi_i \exp(-2\pi i\nu t)\right] = \mathrm{Re}\left[\psi_i b_i\right] = 0.5\left(\psi_i b_i + \psi_i^* b_i^*\right)$

The 2 quadratic terms: $\left\langle E_i^2(x)\right\rangle = \left\langle \mathrm{Re}\left[\psi_i b_i\right]\mathrm{Re}\left[\psi_i b_i\right]\right\rangle = \left\langle 0.5\left(\psi_i b_i + \psi_i^* b_i^*\right)\, 0.5\left(\psi_i b_i + \psi_i^* b_i^*\right)\right\rangle$

$= 0.25\left\langle \left(\psi_i b_i\right)^2 + |\psi_i|^2 |b_i|^2 + |\psi_i|^2 |b_i|^2 + \left(\psi_i^* b_i^*\right)^2 \right\rangle$

$= 0.25\left(\left\langle \left(\psi_i b_i\right)^2\right\rangle + \left\langle |\psi_i|^2 |b_i|^2\right\rangle + \left\langle |\psi_i|^2 |b_i|^2\right\rangle + \left\langle \left(\psi_i^* b_i^*\right)^2\right\rangle\right) = 0.25\left(0 + 2|\psi_i|^2 |b_i|^2 + 0\right)$

(because $\left\langle \left(\psi_i b_i\right)^2 = \left\langle \left(\psi_i^* b_i^*\right)^2\right\rangle = 0\right\rangle$; $|b_i|^2 = 1$) $\to \left\langle E_1^2(x)\right\rangle = 0.50|\psi_1|^2$ and $\left\langle E_2^2(x) = 0.50|\psi_2|^2\right\rangle$

The cross term: $\left\langle E_1(x)\, E_2(x)\right\rangle = \left\langle \mathrm{Re}\left[\psi_1 b_1\right]\mathrm{Re}\left[\psi_2 b_2\right]\right\rangle = \left\langle 0.5\left(\psi_1 b_1 + \psi_1^* b_1^*\right) 0.5\left(\psi_2 b_2 + \psi_2^* b_2^*\right)\right\rangle$

$= 0.25\left\langle \left(\psi_1 b_1 \psi_2 b_2\right) + \left(\psi_1 b_1 \psi_2^* b_2^*\right) + \left(\psi_1^* b_1^* \psi_2 b_2\right) + \left(\psi_1^* b_1^* \psi_2^* b_2^*\right)\right\rangle$

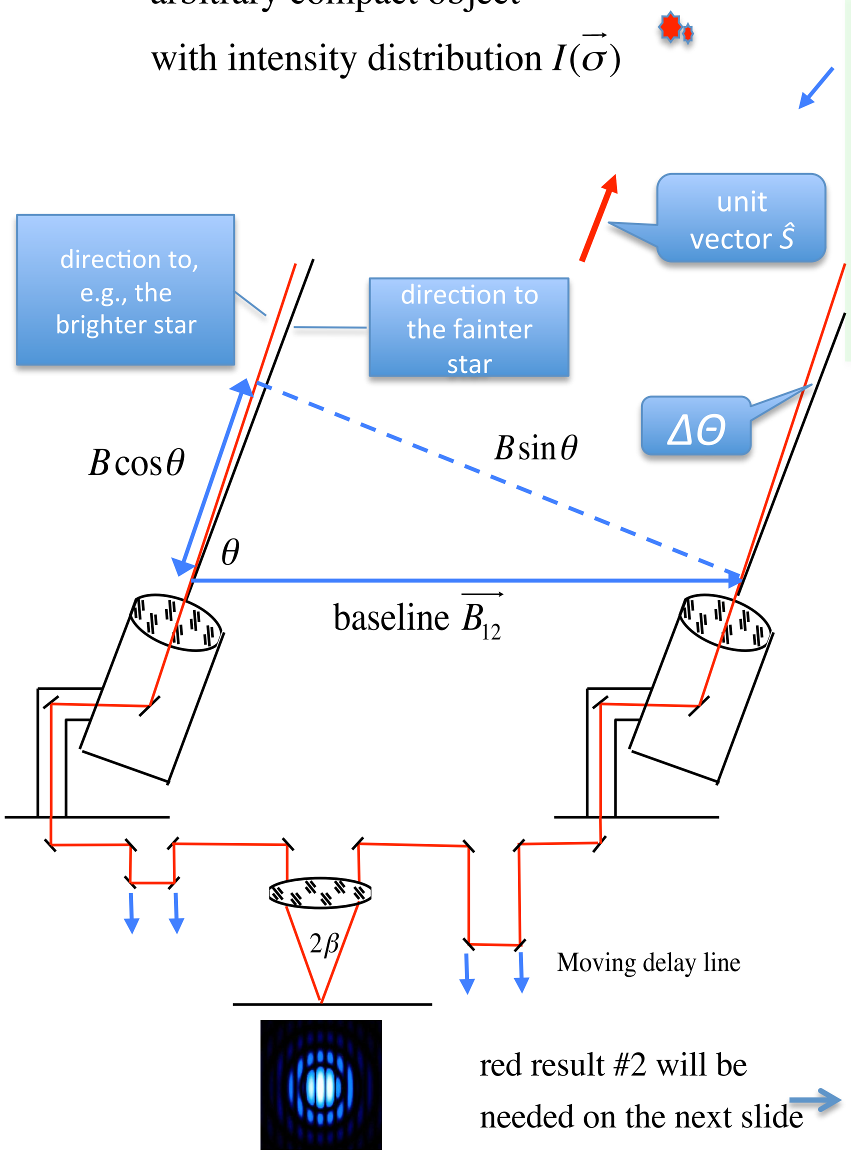
$= 0.25\left(\psi_1 \psi_2 \left\langle b_1 b_2\right\rangle\right) + \psi_1 \psi_2^* \left\langle b_1 b_2^*\right\rangle + \psi_1^* \psi_2 \left\langle b_1^* b_2\right\rangle + \psi_1^* \psi_2^* \left\langle b_1^* b_2^*\right\rangle$

$= 0.25\left(0 + \psi_1 \psi_2^* + \psi_1^* \psi_2 + 0\right)$ (because: $\left\langle b_1 b_2\right\rangle = 0, \left\langle b_1 b_2^*\right\rangle = 1, \left\langle b_1^* b_2\right\rangle = 1, \left\langle b_1^* b_2^*\right\rangle = 0$

Interferogram from an arbitrary object (2)

arbitrary compact object

with intensity distribution $I(\vec{\sigma})$



red result #2 will be needed on the next slide

$\vec{B}_{12}$ denotes the vector baseline between the 2 telescopes,

$\vec{S}$ the unit vector pointing to an element of the object,

$I(\vec{S})$ object intensity in direction $\vec{S}$, and

$I(\vec{S})d\Omega$ the flux from this element within a small solid angle $d\Omega$,

$\vec{\sigma} = (l,m)$ and $\vec{u} = (u,v)$ denote the $2-D$ coordinate vectors

in the object and interferometer baseline coordinate system.

We already derived above $\tau_{ext,12} = \frac{1}{c}B\cos\theta$, which can be written

(vector notation): $\tau_{ext,12} = \frac{1}{c}\vec{B}_{12}\cdot\vec{S}$, where $\vec{B}_{12}$ is the vector baseline.

We assume that the delay line is adjusted to give zero OPD

for an object element in the direction of $\vec{S}_0$ (called phase center).

This is obtained if the internal delay is $\tau_{\mathrm{int},12} = -\frac{1}{c}\vec{B}_{12}\cdot\vec{S}_0$. In this case,

the net delay for a beam from an object element in direction of $\vec{S}$ is

$$\tau_{12} = \left(\frac{1}{c}\vec{B}_{12}\cdot\vec{S}\right) - \left(\frac{1}{c}\vec{B}_{12}\cdot\vec{S}_0\right) = \frac{1}{c}\vec{B}_{12}\cdot\left(\vec{S}-\vec{S}_0\right) = \frac{1}{c}\vec{B}_{12}\cdot\vec{\sigma},$$

where $\vec{\sigma} = \vec{S} - \vec{S}_0$ is the coordinate vector in the object plane and the

phase shift $\phi_{12} = 2\pi\nu\tau_{12} = \tau_{12}c2\pi\frac{1}{\lambda}$ (previous slide; $\nu\lambda = c$ or $\frac{\nu}{c} = \frac{1}{\lambda}$).

Inserting $\tau_{12} \rightarrow 2\pi\nu\tau_{12} = 2\pi\nu\frac{1}{c}\vec{\sigma}\cdot\vec{B}_{12} = 2\pi\frac{1}{\lambda}\vec{\sigma}\cdot\vec{B}_{12} = 2\pi\vec{u}\cdot\vec{\sigma}$,

where $\vec{u} = \vec{B}_{12}/\lambda$ is the vector baseline in units of the wavelength.

Interferogram from an arbitrary object (3)

If we insert $2\pi\nu\tau_{12} = 2\pi\vec{u}\cdot\vec{\sigma} = \phi_{12}$ and $\nu(\tau_{BC,1}(x) - \tau_{BC,2}(x) = sx$ (s = fringe frequeny) into the derived $i(x)$ equation, we obtain

$$i(x) = 2F_0\left\{1 + \mathrm{Re}\left[\exp\left(-2\pi i\vec{u}\cdot\vec{\sigma} + 2\pi isx\right)\right]\right\}$$

The object intensity distribution $I(\vec{\sigma})$ can be represented by a grid of point sources spaced by small distances dl and dm.
The flux from each of these point sources at position $\vec{\sigma}$ is given by $I(\vec{\sigma})\,dl\,dm$.
Each of these point sources generates its own interferogram (with certain brightness and phase shift) and we observe the sum of the intensity distributions of these interferograms.
Therefore, the sum intensity distribution $i(x)$ of the total interferogram of the total object $I(\vec{\sigma})$ (for which we use the same name as for the previous interferograms for simplicity) is the integral $\left(\text{all integrals are } \int_{-\infty}^{+\infty}\right)$

$$i(x) = \iint I(\vec{\sigma})\left[1 + \mathrm{Re}\left\{\exp(2\pi isx)\,\exp(-2\pi i\vec{\sigma}\cdot\vec{u})\right\}\right]dl\,dm \rightarrow$$

$$i(x) = \iint I(\vec{\sigma})\,dl\,dm + \mathrm{Re}\left\{\exp(2\pi isx)\iint I(\vec{\sigma})\exp(-2\pi i\vec{\sigma}\cdot\vec{u})\,dl\,dm\right\}.$$

The last integral

$\iint I(\vec{\sigma})\exp(-2\pi i\vec{\sigma}\cdot\vec{u})\,dl\,dm = F(\vec{u})$ is called coherent or correlated flux $F(\vec{u})$, and $F(\vec{u})$ is the Fourier transform of $I\left(\vec{\sigma}\right)$.

$\iint I(\vec{\sigma})\,dl\,dm = F(0) =$ coherent flux at baseline length zero is equal to the total object flux (or zero-spacing flux).

Complex visibility, visibility modulus, and van Cittert-Zernike theorem

$\rightarrow i(x) = F(0) + \mathrm{Re}\left\{F(\vec{u})\exp(2\pi isx)\right\}$

$\rightarrow i(x) = F(0) + \mathrm{Re}\left\{\left|F(\vec{u})\right|\exp\left[i\left(2\pi sx + \phi_F\right)\right]\right\}$ with $F(\vec{u}) = \left|F(\vec{u})\right|\exp\left(i\phi_F\right) = \iint I(\vec{\sigma})(\exp(-2\pi i\vec{\sigma}\cdot\vec{u})dl\,dm$

$\rightarrow i(x) = F(0) + \left|F(\vec{u})\right|\cos\left(2\pi sx + \phi_F\right)$

Therefore, the correlated flux $F(\vec{u}) = \left|F(\vec{u})\right|\exp\left(i\phi_F\right) = FT\left[I(\vec{\sigma})\right]$ influences both

(1) the contrast of the fringes of the interferogram $i(x)$ because of the $\left|F(\vec{u})\right|$ term in front of the cosine and

(2) the phase of the fringes of $i(x)$ because of the object Fourier phase ϕ_F in the cosine.

The normalised correlated flux of $F(\vec{u})$, i.e., $\frac{F(\vec{u})}{F(0)}$, is called the **complex** visibility $V(\vec{u}) = \frac{F(\vec{u})}{F(0)}$.

$F(\vec{u})$ is an important quantity because the normalisation factor $F(0)$ is often difficult to determine.

$\left|V\right| = \frac{i_{\max} - i_{\min}}{i_{\max} + i_{\min}}$ (= fringe contrast) is called fringe visibility, visibility amplitude, or visibility modulus.

$i_{\max}$ and $i_{\min}$ are the maximum and minimum intensities in the fringe pattern.

$\rightarrow V(\vec{u}) = \frac{F(\vec{u})}{F(0)} = \iint I'(\vec{\sigma})\exp(-2\pi i\vec{\sigma}\cdot\vec{u})dl\,dm,\quad$ where $I'\left(\vec{\sigma}\right) = \frac{I(\vec{\sigma})}{\iint I(\vec{\sigma})dl\,dm}$

This Fourier transform relation between the complex visibility $V(\vec{u})$ and the intensity distribution $I(\vec{\sigma})$ of the object is called the van Cittert-Zernike theorem.

Derivation of the Convolution Theorem

$$\hat{F}\left[\int_{(\xi)} g(\xi)h(x-\xi)\,d\xi\right] = \int_{(x)}\left[\int_{(\xi)} g(\xi)h(x-\xi)\,d\xi\right]\exp(-2\pi i f_x x)\,dx$$

$$= \int_{(\xi)} g(\xi)\quad\left[\int_{(x)} h(x-\xi)\exp(-2\pi i f_x x)\,dx\right]d\xi$$

(with $x' = x-\xi$ or $x = x'+\xi$ follows)

$$= \int_{(\xi)} g(\xi)\quad\int_{(x')} h(x')\exp\left[-2\pi i f_x (x'+\xi)\right]dx'\quad d\xi$$

$$= \int_{(\xi)} g(\xi)\exp\left[-2\pi i f_x \xi\right]d\xi\quad\int_{(x')} h(x')\exp\left[-2\pi i f_x x'\right]dx'$$

$$= \hat{F}\left[g(x)\right]\ \hat{F}\left[h(x)\right] = G(f_x)\,H(f_x)$$

So, $\hat{F}(g\otimes h) = \hat{F}(g)\cdot\hat{F}(h)$.

Derivation of the autocorrelation theorem (1-D)

$$\hat{F}\left[\int_{(\xi)} g(\xi)\, g^*(\xi - x)\, d\xi\right] = \hat{F}\left[\int_{(\xi')} g(\xi' + x)\, g^*(\xi')\, d\xi'\right] = \qquad \left(\text{with } \xi' = \xi - x\right)$$

$$= \int_{(\xi')}\int_{(x)} g(\xi' + x)\, g^*(\xi') \exp(-2\pi i f_x x)\, dx\, d\xi'$$

$$= \int_{(\xi')} g^*(\xi') \left[\int_{(x)} g(\xi' + x) \exp(-2\pi i f_x x)\, dx\right] d\xi'$$

$$= \int_{(\xi')} g^*(\xi')\; G(f_x) \exp(+2\pi i f_x \xi')\, d\xi' \quad \text{(shift theorem)}$$

$$= \int_{(\xi')} g^*(\xi') \exp(+2\pi i f_x \xi')\, d\xi' \;\; G(f_x)$$

$$= G^*(f_x)\, G(f_x) = \left|G(f_x)\right|^2. \quad \text{So:} \quad \hat{F}\left[\hat{A}(g)\right] = \left|G(f_x)\right|^2 \quad \left(\hat{A} = \text{autocorrelation operator}\right)$$

Derivation of the Shift Theorem (1-D)

$$\hat{F}\left[g(x-a)\right]$$

$$= \int_{(x)} g(x-a)\exp\left[-2\pi i f_x x\right]dx$$

$$\text{(with } x' = x - a, x = x' + a\text{)}$$

$$= \int_{(x')} g(x')\exp\left[-2\pi i f_x (x'+a)\right]dx' = \int_{(x')} g(x')\exp\left[-2\pi i f_x x'\right] \quad \exp\left[-2\pi i f_x a\right]dx'$$

$$= \exp\left[-2\pi i f_x a\right]\int_{(x')} g(x')\exp\left[-2\pi i f_x x'\right]dx'$$

$$= \exp\left[-2\pi i f_x a\right]G(f_x)$$